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Binomial Transform of the Generalized Guglielmo Sequence

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ABSTRACT
In this paper, we define the binomial transform of the generalized Guglielmo sequence and as special cases, the binomial transform of the triangular, Lucas-triangular, oblong, and pentagonal sequences will be introduced. We investigate their properties in detail. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these binomial transforms. Moreover, we give some identities such as Catalan's identity, Cassani's identity, and matrices related to these binomial transforms.

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KEYWORDS
Binomial transform, Triangular numbers, Lucas-Triangular numbers, Oblong numbers, Pentagonal numbers, Binomial transform of generalized Guglielmo numbers.

Introduction

In this paper, we introduce the binomial transform of the generalized Guglielmo sequence and we investigate, in detail, four special cases named the binomial transform of the triangular, Lucas-triangular, oblong and pentagonal sequences. We investigate their properties in the next sections. In this section, we present some properties of the generalized Guglielmo sequence that studied by Soykan [0.2].

A generalized Guglielmo sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is given by the third-order recurrence relations.

$$W_n = 3W_{n-1} - 3W_{n-2} + W_{n-3} \tag{0.1}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 3W_{-(n-1)} - 3W_{-(n-2)} + W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (0.1) holds for all integer n.

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Now we present four special cases of the sequence $\{W_n\}$. Triangular sequence $\{T_n\}_{n\geq 0}$,triangular-Lucas sequence $\{H_n\}_{n\geq 0}$, oblong sequence $\{O_n\}_{n\geq 0}$ and pentagonal sequence $\{p_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, \quad T_0 = 0,\ T_1 = 1,\ T_2 = 3, \tag{0.2}$$
$$H_n = 3H_{n-1} - 3H_{n-2} + H_{n-3}, \quad H_0 = 3,\ H_1 = 3,\ H_2 = 3, \tag{0.3}$$
$$O_n = 3O_{n-1} - 3O_{n-2} + O_{n-3}, \quad O_0 = 0,\ O_1 = 2,\ O_2 = 6, \tag{0.4}$$
$$p_n = 3p_{n-1} - 3p_{n-2} + p_{n-3}, \quad p_0 = 0,\ p_1 = 1,\ p_2 = 5. \tag{0.5}$$

The sequences $\{T_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$, $\{O_n\}_{n\geq 0}$ and $\{p_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
T_{-n} &= 3T_{-(n-1)} - 3T_{-(n-2)} + T_{-(n-3)},\\
H_{-n} &= 3H_{-(n-1)} - 3H_{-(n-2)} + H_{-(n-3)},\\
O_{-n} &= 3O_{-(n-1)} - 3O_{-(n-2)} + O_{-(n-3)},\\
p_{-n} &= 3p_{-(n-1)} - 3p_{-(n-2)} + p_{-(n-3)},
\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (0.2)-(0.5) hold for all integer n. Now, we give some properties related to generalized Guglielmo numbers that we need for the rest of the study.

- The Binet formula of generalized Guglielmo numbers. Binet formula of generalized Guglielmo numbers can be given as

$$W_n = A_1 + A_2 n + A_3 n^2 \tag{0.6}$$

where

$$\begin{aligned}
A_1 &= W_0,\\
A_2 &= \frac{1}{2}(-W_2 + 4W_1 - 3W_0),\\
A_3 &= \frac{1}{2}(W_2 - 2W_1 + W_0),
\end{aligned}$$

i.e.,

$$W_n = W_0 + \frac{1}{2}(-W_2 + 4W_1 - 3W_0)n + \frac{1}{2}(W_2 - 2W_1 + W_0)n^2. \tag{0.7}$$

- For all integers n, triangular, triangular-Lucas, oblong and pentagonal numbers (using initial conditions in (0.7)) can be expressed using Binet's formulas as

$$\begin{aligned}
T_n &= \frac{n(n+1)}{2},\\
H_n &= 3,\\
O_n &= n(n+1),\\
p_n &= \frac{1}{2}n\left(3n-1\right),
\end{aligned}$$

respectively.

- Suppose that $f_{W_n}(x) = \sum\limits_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized Guglielmo sequence $\{W_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} W_n x^n$ is given by

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 3W_0)x + (W_2 - 3W_1 + 3W_0)x^2}{1 - 3x + 3x^2 - x^3}. \tag{0.8}$$

- Here, the characteristic equation of the Generalized Guglielmo sequence

$$x^3 - 3x^2 + 3x - 1 = 0.$$

- (Simpson's formula for generalized Guglielmo numbers)For all integers n, we have

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = -(W_2 - 2W_1 + W_0)^3.$$

For more detail, see [0.2].

Binomial Transform of the Generalized Guglielmo Sequence

In [0.2, p. 137], Knuth defined the idea of the binomial transform. Given a sequence of numbers (a_n), its binomial transform $(\hat{a}_n)$ defined as follows

$$\hat{a}_n = \sum_{i=0}^{n} \binom{n}{i} a_i, \quad \text{with inversion } a_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \hat{a}_i,$$

or, in the symmetric version

$$\hat{a}_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{i+1} a_i, \quad \text{with inversion } a_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{i+1} \hat{a}_i.$$

For more information on binomial transform, see, [0.2,0.2,0.2,0.2].

In this section, we define the binomial transform of the generalized Guglielmo sequence W_n and as special cases the binomial transform of the triangular, Lucas-triangular, oblong and pentagonal sequences.

Definition 0.1. *The binomial transform of the generalized Guglielmo sequence W_n is defined by*

$$b_n = \widehat{W}_n = \sum_{i=0}^{n} \binom{n}{i} W_i.$$

The some terms of b_n may be given as

$$\begin{aligned} b_0 &= \sum_{i=0}^{0} \binom{0}{i} W_i = W_0, \\ b_1 &= \sum_{i=0}^{1} \binom{1}{i} W_i = W_0 + W_1, \\ b_2 &= \sum_{i=0}^{2} \binom{2}{i} W_i = W_0 + 2W_1 + W_2. \end{aligned}$$

If we translate the b_n to matrix form that includes lower-triangular matrix, we get

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \\ W_4 \\ \vdots \end{pmatrix}.$$

As special cases of $b_n = \widehat{W}_n$, the binomial transforms of the triangular, Lucas-triangular, oblong and pentagonal sequences are defined as follows: The binomial transform of the triangular sequence T_n is

$$\widehat{T}_n = \sum_{i=0}^{n} \binom{n}{i} T_i,$$

the binomial transform of the Lucas-triangular sequence H_n is

$$\widehat{H}_n = \sum_{i=0}^{n} \binom{n}{i} H_i,$$

the binomial transform of the oblong sequence O_n is

$$\widehat{O}_n = \sum_{i=0}^{n} \binom{n}{i} O_i,$$

the binomial transform of the pentagonal sequence p_n is

$$\widehat{p}_n = \sum_{i=0}^{n} \binom{n}{i} p_i.$$

Lemma 0.1.1. *For $n \geq 0$, the binomial transform of the generalized Guglielmo sequence W_n satisfies the following relation:*

$$b_{n+1} = \sum_{i=0}^{n} \binom{n}{i} (W_i + W_{i+1}).$$

Proof. We use the following well-known identity:

$$\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}.$$

Note also that

$$\binom{n+1}{0} = \binom{n}{0} = 1 \text{ and } \binom{n}{n+1} = 0.$$

Then

$$\begin{aligned}
b_{n+1} &= W_0 + \sum_{i=1}^{n+1} \binom{n+1}{i} W_i \\
&= W_0 + \sum_{i=1}^{n+1} \binom{n}{i} W_i + \sum_{i=1}^{n+1} \binom{n}{i-1} W_i \\
&= W_0 + \sum_{i=1}^{n} \binom{n}{i} W_i + \sum_{i=0}^{n} \binom{n}{i} W_{i+1} \\
&= \sum_{i=0}^{n} \binom{n}{i} W_i + \sum_{i=0}^{n} \binom{n}{i} W_{i+1} \\
&= \sum_{i=0}^{n} \binom{n}{i} (W_i + W_{i+1}).
\end{aligned}$$

This completes the proof. □

Remark 0.2. *From the last Lemma, we see that*

$$b_{n+1} = b_n + \sum_{i=0}^{n} \binom{n}{i} W_{i+1}.$$

The following theorem gives recurrent relations of the binomial transform of the generalized Tribonacci sequence.

Theorem 0.3. *For $n \geq 0$, the binomial transform of the generalized Guglielmo sequence W_n satisfies the following recurrence relation:*

$$b_{n+3} = 6b_{n+2} - 12b_{n+1} + 8b_n. \tag{0.9}$$

Proof. To show (0.9), writing

$$b_{n+3} = r_1 \times b_{n+2} + r_2 \times b_{n+1} + r_3 \times b_n$$

and taking the values $n = 0, 1, 2$ and then solving the system of equations

$$\begin{aligned} b_3 &= r_1 \times b_2 + r_2 \times b_1 + r_3 \times b_0, \\ b_4 &= r_1 \times b_3 + r_2 \times b_2 + r_3 \times b_1, \\ b_5 &= r_1 \times b_4 + r_2 \times b_3 + r_3 \times b_2. \end{aligned}$$

Hence, we find that $r_3 = 6, r_3 = -12, r_3 = 8$. □

The sequence $\{b_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$b_{-n} = \frac{3}{2}b_{-n+1} - \frac{3}{4}b_{-n+2} + \frac{1}{8}b_{-n+3}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (0.9) holds for all integer n.

Note that the recurrence relation (0.9) is independent from initial values. So,

$$\begin{aligned} \widehat{T}_{n+3} &= 6\widehat{T}_{n+2} - 12\widehat{T}_{n+1} + 8\widehat{T}_n, \\ \widehat{H}_{n+3} &= 6\widehat{H}_{n+2} - 12\widehat{H}_{n+1} + 8\widehat{H}_n, \\ \widehat{O}_{n+3} &= 6\widehat{O}_{n+2} - 12\widehat{O}_{n+1} + 8\widehat{O}_n, \\ \widehat{p}_{n+3} &= 6\widehat{p}_{n+2} - 12\widehat{p}_{n+1} + 8\widehat{p}_n. \end{aligned}$$

The first few terms of the binomial transform of the generalized Guglielmo sequence with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few terms of binomial transform of the generalized Guglielmo sequence.

n	b_n	b_{-n}
0	W_0	
1	$W_0 + W_1$	$\frac{7}{8}W_0 - \frac{1}{2}W_1 + \frac{1}{8}W_2$
2	$W_0 + 2W_1 + W_2$	$\frac{11}{16}W_0 - \frac{5}{8}W_1 + \frac{3}{16}W_2$
3	$2W_0 + 6W_2$	$\frac{1}{2}W_0 - \frac{9}{16}W_1 + \frac{3}{16}W_2$
4	$8W_0 - 16W_1 + 24W_2$	$\frac{11}{32}W_0 - \frac{7}{16}W_1 + \frac{5}{32}W_2$
5	$32W_0 - 80W_1 + 80W_2$	$\frac{29}{128}W_0 - \frac{5}{16}W_1 + \frac{15}{128}W_2$
6	$112W_0 - 288W_1 + 240W_2$	$\frac{37}{256}W_0 - \frac{27}{128}W_1 + \frac{21}{256}W_2$
7	$352W_0 - 896W_1 + 672W_2$	$\frac{23}{256}W_0 - \frac{35}{256}W_1 + \frac{7}{128}W_2$
8	$1024W_0 - 2560W_1 + 1792W_2$	$\frac{7}{128}W_0 - \frac{11}{128}W_1 + \frac{9}{256}W_2$
9	$2816W_0 - 6912W_1 + 4608W_2$	$\frac{67}{2048}W_0 - \frac{27}{512}W_1 + \frac{45}{2048}W_2$
10	$7424W_0 - 17\,920W_1 + 11\,520W_2$	$\frac{79}{4096}W_0 - \frac{65}{2048}W_1 + \frac{55}{4096}W_2$
11	$18\,944W_0 - 45\,056W_1 + 28\,160W_2$	$\frac{23}{2048}W_0 - \frac{77}{4096}W_1 + \frac{33}{4096}W_2$
12	$47\,104W_0 - 110\,592W_1 + 67\,584W_2$	$\frac{53}{8192}W_0 - \frac{45}{4096}W_1 + \frac{39}{8192}W_2$
13	$114\,688W_0 - 266\,240W_1 + 159\,744W_2$	$\frac{121}{32\,768}W_0 - \frac{13}{2048}W_1 + \frac{91}{32\,768}W_2$

The first few terms of the binomial transform numbers of triangulas, Lucas-triangular, oblong and pentegonal sequences with positive subscript and negative subscript are given in the following Table 2.

Table 2. A few terms of four special cases of binomial transform of generalized Guglielmo numbers.

n	0	1	2	3	4	5	6	7	8	9	10	11
$\widehat{T}_n$	0	1	5	18	56	160	432	1120	2816	6912	16640	39424
$\widehat{T}_{-n}$		$-\frac{1}{8}$	$-\frac{1}{16}$	0	$\frac{1}{32}$	$\frac{5}{128}$	$\frac{9}{256}$	$\frac{7}{256}$	$\frac{5}{256}$	$\frac{27}{2048}$	$\frac{35}{4096}$	$\frac{11}{2048}$
$\widehat{H}_n$	3	6	12	24	48	96	192	384	768	1536	3072	6144
$\widehat{H}_{-n}$		$\frac{3}{2}$	$\frac{3}{4}$	$\frac{3}{8}$	$\frac{3}{16}$	$\frac{3}{32}$	$\frac{3}{64}$	$\frac{3}{128}$	$\frac{3}{256}$	$\frac{3}{512}$	$\frac{3}{1024}$	$\frac{3}{2048}$
$\widehat{O}_n$	0	2	10	36	112	320	864	2240	5632	13824	33280	78848
$\widehat{O}_{-n}$		$-\frac{1}{4}$	$-\frac{1}{8}$	0	$\frac{1}{16}$	$\frac{5}{64}$	$\frac{9}{128}$	$\frac{7}{128}$	$\frac{5}{128}$	$\frac{27}{1024}$	$\frac{35}{2048}$	$\frac{11}{1024}$
$\widehat{p}_n$	0	1	7	30	104	320	912	2464	6400	16128	39680	95744
$\widehat{p}_{-n}$		$\frac{1}{8}$	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{11}{32}$	$\frac{35}{128}$	$\frac{51}{256}$	$\frac{35}{256}$	$\frac{23}{256}$	$\frac{117}{2048}$	$\frac{145}{4096}$	$\frac{11}{512}$

Now, we define the Binet's formula of the binomal transform of the generalized Guglielmo sequence.

Theorem 0.4. *For any integer n the Binet's formula of the binomial transform of the generalized Guglielmo sequence is given as*

$$b_n = \widehat{W}_n = (A_1 + A_2\frac{n}{2} + A_3\frac{n(n+1)}{4})2^n \tag{0.10}$$

where

$$\begin{aligned} A_1 &= W_0, \\ A_2 &= \frac{1}{2}(-W_2 + 4W_1 - 3W_0), \\ A_3 &= \frac{1}{2}(W_2 - 2W_1 + W_0). \end{aligned}$$

Proof. For the proof, we use the following identities.

$$\begin{aligned} \sum_{i=0}^{n}\binom{n}{i} &= 2^n, \\ \sum_{i=0}^{n}\binom{n}{i}i &= 2^{n-1}n, \\ \sum_{i=0}^{n}\binom{n}{i}i^2 &= 2^{n-2}n(n+1). \end{aligned}$$

Using (0.6), we can write the b_n as

$$\begin{aligned} b_n &= \sum_{i=0}^{n}\binom{n}{i}W_i \\ &= \sum_{i=0}^{n}\binom{n}{i}(A_1 + A_2 i + A_3 i^2) \\ &= A_1\sum_{i=0}^{n}\binom{n}{i} + A_2\sum_{i=0}^{n}\binom{n}{i}i + A_3\sum_{i=0}^{n}\binom{n}{i}i^2. \end{aligned}$$

Consequently, the proof can be done. □

For all integers n, using Theorem 0.4, Binet's formulas of binomial transforms of the triangular, Lucas-triangular, oblong and pentagonal numbers are given in the following corollary, respectively.

Corollary 0.5. *The Binet's formula of binomial transforms of the triangular, Lucas-triangular, oblong, and pentagonal numbers are given as follows.*

(a) $\widehat{T}_n = n\,(n+3) \times 2^{n-3}$.
(b) $\widehat{H}_n = 3 \times 2^{n}$.
(c) $\widehat{O}_n = n\,(n+3) \times 2^{n-2}$.
(d) $\widehat{p}_n = n\,(3n+1) \times 2^{n-3}$.

Obtaining Binet Formula of Binomial Transform of Generalized Guglielmo Sequence From Generating Function

The generating function of the binomial transform of the generalized Guglielmo sequence W_n is a power series centered at the origin whose coefficients are the binomial transform of the generalized Guglielmo sequence.

Next, we give the ordinary generating function $f_{b_n}(x) = \sum\limits_{n=0}^{\infty} b_n x^n$ of the sequence b_n.

Theorem 0.6. *Suppose that $f_{b_n}(x) = \sum\limits_{n=0}^{\infty} b_n x^n$ is the ordinary generating function of the binomial transform of the generalized Guglielmo sequence $\{W_n\}_{n\geq 0}$. Then, $f_{b_n}(x)$ is given by*

$$f_{b_n}(x) = \frac{b_0 + (b_1 - 6b_0)x + (b_2 - 6b_1 + 12b_0)x^2}{1 - 6x + 12x^2 - 8x^3}. \tag{0.11}$$

Proof. Using the definition of the binomial transform of the generalized Guglielmo sequence, we obtain

$$\begin{aligned}
(1-6x+12x^2-8x^3)f_{b_n}(x) &= \sum_{n=0}^{\infty} b_n x^n - 6x\sum_{n=0}^{\infty} b_n x^n + 12x^2\sum_{n=0}^{\infty} b_n x^n - 8x^3\sum_{n=0}^{\infty} b_n x^n \\
&= \sum_{n=0}^{\infty} b_n x^n - 6\sum_{n=0}^{\infty} b_n x^{n+1} + 12\sum_{n=0}^{\infty} b_n x^{n+2} - 8\sum_{n=0}^{\infty} b_n x^{n+3} \\
&= \sum_{n=0}^{\infty} b_n x^n - 6\sum_{n=1}^{\infty} b_{n-1} x^{n} + 12\sum_{n=2}^{\infty} b_{n-2} x^{n} - 8\sum_{n=3}^{\infty} b_{n-3} x^{n} \\
&= b_0 + b_1x^1 + b_2x^2 - 6b_0x^1 - 6b_1x^2 + 12b_0x^2 \\
&= +\sum_{n=3}^{\infty}(b_n - 6b_{n-1} + 12b_{n-2} - 8b_{n-3})x^n \\
&= b_0 + (b_1 - 6b_0)x + (b_2 - 6b_1 + 12b_0)x^2.
\end{aligned}$$

Then rearranging the above equation, we get (0.11). □

If we take,

$$\begin{aligned} b_0 &= W_0, \\ b_1 &= W_0 + W_1, \\ b_2 &= W_0 + 2W_1 + W_2. \end{aligned}$$

(0.11) is written as

$$f_{b_n}(x) = \frac{W_0 + (W_1 - 5W_0)x + (7W_0 - 4W_1 + W_2)x^2}{1 - 6x + 12x^2 - 8x^3}$$

Proposition 1. *P. Barry shows in [0.2] that if $A(x)$ is the generating function of the sequence $\{a_n\}$, then*

$$S(x) = \frac{1}{1-x}A(\frac{x}{1-x})$$

is the generating function of the sequence $\{b_n\}$ with $b_n = \sum\limits_{i=0}^{n} \binom{n}{i} a_i$.

Note that, in our case, using Proposition 1 and (0.8), we get

$$\begin{aligned} S(x) &= \frac{1}{1-x}\frac{W_0 + (W_1 - 3W_0)(\frac{x}{1-x}) + (W_2 - 3W_1 + 3W_0)(\frac{x}{1-x})^2}{1 - 3(\frac{x}{1-x}) + 3(\frac{x}{1-x})^2 - (\frac{x}{1-x})^3} \\ &= \frac{W_0 + (W_1 - 5W_0)x + (7W_0 - 4W_1 + W_2)x^2}{1 - 6x + 12x^2 - 8x^3}. \end{aligned}$$

From Theorem 0.6, we get the following corollary.

Corollary 0.7. *Generating functions of the binomial transform of the triangular, Lucas-triangular, oblong and pentagonal numbers are*

$$\begin{aligned} \sum_{n=0}^{\infty} \widehat{T}_n x^n &= \frac{x - x^2}{1 - 6x + 12x^2 - 8x^3}, \\ \sum_{n=0}^{\infty} \widehat{H}_n x^n &= \frac{3 - 12x + 12x^2}{1 - 6x + 12x^2 - 8x^3}, \\ \sum_{n=0}^{\infty} \widehat{O}_n x^n &= \frac{2x - 2x^2}{1 - 6x + 12x^2 - 8x^3}, \\ \sum_{n=0}^{\infty} \widehat{p}_n x^n &= \frac{x + x^2}{1 - 6x + 12x^2 - 8x^3}, \end{aligned}$$

respectively.

We next find Binet's formula of the Binomial transform of the generalized Guglielmo numbers $\{W_n\}$ by the use of generating function for b_n.

Proposition 2. *(Examples of generating functions for simple sequence [0.2])The following equality is true*

$$\sum_{n=0}^{\infty} \alpha^n \binom{n+k}{k} x^n = \frac{1}{(1-\alpha x)^{k+1}}.$$

Next, we obtain the Binet's formula, given in Theorem 0.4, by the generating function of the binomial transform of the generalized Guglielmo sequence.

Theorem 0.8. *Binet's formula of the Binomial transform of the generalized Guglielmo numbers are*

$$b_n = b_n = (A_1 + A_2 \frac{n}{2} + A_3 \frac{n(n+1)}{4})2^n \tag{0.12}$$

where

$$\begin{aligned} A_1 &= W_0, \\ A_2 &= \frac{1}{2}(-W_2 + 4W_1 - 3W_0), \\ A_3 &= \frac{1}{2}(W_2 - 2W_1 + W_0). \end{aligned}$$

Proof. Using (*0.11*), we get the following identity

$$\begin{aligned} \sum_{n=0}^{\infty} b_n x^n &= \frac{b_0 + (b_1 - 6b_0)x + (b_2 - 6b_1 + 12b_0)x^2}{1 - 6x + 12x^2 - 8x^3} \\ &= \frac{d_1}{(1-2x)} + \frac{d_2}{(1-2x)^2} + \frac{d_3}{(1-2x)^3}. \end{aligned}$$

Then, we get

$$\begin{aligned} b_0 &= -d_2 - d_3 - d_1, \\ b_1 - 6b_0 &= 4d_1 + 2d_2, \\ b_2 - 6b_1 + 12b_0 &= -4d_1. \end{aligned}$$

Hence, solving the above system of equations and using Proposition 2, we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_n x^n &= \frac{\frac{1}{4}(12b_0 - 6b_1 + b_2)}{(1-2x)} + \frac{-\frac{1}{2}(6b_0 - 5b_1 + b_2)}{(1-2x)^2} + \frac{\frac{1}{4}(4b_0 - 4b_1 + b_2)}{(1-2x)^3} \\ &= \sum_{n=0}^{\infty} (\frac{1}{4}(12b_0 - 6b_1 + b_2) 2^n - \frac{1}{2}(6b_0 - 5b_1 + b_2)(n+1)2^n + \\ &= \frac{1}{8}(4b_0 - 4b_1 + b_2)(n^2 + 3n + 2)2^n)x^n. \end{aligned}$$

Thus, we get

$$b_n = (\frac{1}{4}(12b_0 - 6b_1 + b_2)2^n - \frac{1}{2}(6b_0 - 5b_1 + b_2)(n+1)2^n + \frac{1}{8}(4b_0 - 4b_1 + b_2)(n^2+3n+2)2^n).$$

Consequently, If we take $b_0 = W_0$, $b_1 = W_0 + W_1$ and $b_2 = W_0 + 2W_1 + W_2$ we get the following identity

$$b_n = (A_1 + A_2\frac{n}{2} + A_3\frac{n(n+1)}{4})2^n$$

where A_1, A_2 and A_3 are stated in the theorem. □

Simson's Formulas

It's well known that many authors studied the Simpson Formulas for different sequences. Similarly, in this section, we introduce the Simson formula for binomial transforms of generalized Guglielmo numbers.

Theorem 0.9. *For all integers n, Simson formula of binomial transforms of generalized Guglielmo numbers are given as*

$$\begin{vmatrix} b_{n+2} & b_{n+1} & b_n \\ b_{n+1} & b_n & b_{n-1} \\ b_n & b_{n-1} & b_{n-2} \end{vmatrix} = 8^n \begin{vmatrix} b_2 & b_1 & b_0 \\ b_1 & b_0 & b_{-1} \\ b_0 & b_{-1} & b_{-2} \end{vmatrix}. \tag{0.13}$$

Proof. For the proof, we use the mathematical induction on n. First, we assume that $n \geq 0$. If we take $n = 0$, it's easily seen that (0.13) is hold. Let (0.13) is true for $n = k$ so we can write the following identity.

$$\begin{vmatrix} b_{k+2} & b_{k+1} & b_k \\ b_{k+1} & b_k & b_{k-1} \\ b_k & b_{k-1} & b_{k-2} \end{vmatrix} = 8^k \begin{vmatrix} b_2 & b_1 & b_0 \\ b_1 & b_0 & b_{-1} \\ b_0 & b_{-1} & b_{-2} \end{vmatrix}.$$

Then, we will show that (0.13) is true for $n = k + 1$.

$$\begin{aligned}
\begin{vmatrix} b_{k+3} & b_{k+2} & b_{k+1} \\ b_{k+2} & b_{k+1} & b_k \\ b_{k+1} & b_k & b_{k-1} \end{vmatrix} &= \begin{vmatrix} 6b_{k+2} - 12b_{k+1} + 8b_k & b_{k+2} & b_{k+1} \\ 6b_{k+1} - 12b_k + 8b_{k-1} & b_{k+1} & b_k \\ 6b_k - 12b_{k-1} + 8b_{k-2} & b_k & b_{k-1} \end{vmatrix} \\
&= 6\begin{vmatrix} b_{k+2} & b_{k+2} & b_{k+1} \\ b_{k+1} & b_{k+1} & b_k \\ b_k & b_k & b_{k-1} \end{vmatrix} - 12\begin{vmatrix} b_{k+1} & b_{k+2} & b_{k+1} \\ b_k & b_{k+1} & b_k \\ b_{k-1} & b_k & b_{k-1} \end{vmatrix} \\
&\quad +8\begin{vmatrix} b_k & b_{k+2} & b_{k+1} \\ b_{k-1} & b_{k+1} & b_k \\ b_{k-2} & b_k & b_{k-1} \end{vmatrix} \\
&= 8^{k+1}\begin{vmatrix} b_2 & b_1 & b_0 \\ b_1 & b_0 & b_{-1} \\ b_0 & b_{-1} & b_{-2} \end{vmatrix}.
\end{aligned}$$

Note that, for the case $n < 0$ the proof can be done similarly. Thus, the proof is completed. □

The previous theorem gives the following corollary.

Corollary 0.10. *For all integers n, the following identities are true.*

(a) $\begin{vmatrix} \widehat{T}_{n+2} & \widehat{T}_{n+1} & \widehat{T}_{n} \\ \widehat{T}_{n+1} & \widehat{T}_{n} & \widehat{T}_{n-1} \\ \widehat{T}_{n} & \widehat{T}_{n-1} & \widehat{T}_{n-2} \end{vmatrix} = -2^{3n-6}.$

(b) $\begin{vmatrix} \widehat{H}_{n+2} & \widehat{H}_{n+1} & \widehat{H}_{n} \\ \widehat{H}_{n+1} & \widehat{H}_{n} & \widehat{H}_{n-1} \\ \widehat{H}_{n} & \widehat{H}_{n-1} & \widehat{H}_{n-2} \end{vmatrix} = 0.$

(c) $\begin{vmatrix} \widehat{O}_{n+2} & \widehat{O}_{n+1} & \widehat{O}_{n} \\ \widehat{O}_{n+1} & \widehat{O}_{n} & \widehat{O}_{n-1} \\ \widehat{O}_{n} & \widehat{O}_{n-1} & \widehat{O}_{n-2} \end{vmatrix} = -2^{3n-3}.$

(d) $\begin{vmatrix} \widehat{p}_{n+2} & \widehat{p}_{n+1} & \widehat{p}_{n} \\ \widehat{p}_{n+1} & \widehat{p}_{n} & \widehat{p}_{n-1} \\ \widehat{p}_{n} & \widehat{p}_{n-1} & \widehat{p}_{n-2} \end{vmatrix} = -27 \times 2^{3n-6}.$

Some Identites

We now present a few special identities for the binomial transforms of generalized Guglielmo numbers. The following Theorem presents the Catalan's identity for the binomial transforms of generalized Guglielmo numbers.

Theorem 0.11. *(Catalan's identity) For all integers n and m, the following identity holds.*

$$b_{n+m}b_{n-m}-b_n^2 = -2^{2n-4}m^2(-m^2A_3^2+2n^2A_3^2+4nA_2A_3+2nA_3^2+4A_2^2+4A_2A_3+A_3^2-8A_1A_3).$$

Proof. The proof has been seen easily using (0.10).

As special cases of the above theorem, we give Catalan's identity of the binomial transforms of generalized Guglielmo numbers. Firstly, we present Catalan's identity of the binomial transforms of triangular numbers.

Corollary 0.12. *(Catalan's identity of the binomial transforms of triangular numbers) For all integers n and m, the following identity holds.*

$$\widehat{T}_{n+m}\widehat{T}_{n-m} - \widehat{T}_n^2 = 2^{2n-6}m^2\left(-6n + m^2 - 2n^2 - 9\right).$$

Proof. Taking $b_n = \widehat{T}_n$ in Theorem 0.11 we get the required result. □

Next, we present the Catalan's identity for the binomial transforms of generalized Lucas-triangular numbers.

Corollary 0.13. *(Catalan's identity for the binomial transforms of Lucas-triangular numbers) For all integers n and m, the following identity holds.*

$$\widehat{H}_{n+m}\widehat{H}_{n-m} - \widehat{H}_n^2 = 0.$$

Proof. Taking $b_n = \widehat{H}_n$ in Theorem 0.11 we get the required result. □

Next, we present the Catalan's identity for the binomial transforms of generalized oblong numbers.

Corollary 0.14. *(Catalan's identity for the binomial transforms of oblong numbers) For all integers n and m, the following identity holds.*

$$\widehat{O}_{n+m}\widehat{O}_{n-m}-\widehat{O}_n^2=2^{2n-4}m^2\left(-6n+m^2-2n^2-9\right).$$

Proof. Taking $b_n=\widehat{O}_n$ in Theorem 0.11 we get the required result. □

Next, we present the Catalan's identity for the binomial transforms of generalized pentagonal numbers.

Corollary 0.15. *(Catalan's identity for the binomial transforms of pentagonal numbers) For all integers n and m, the following identity holds.*

$$\widehat{p}_{n+m}\widehat{p}_{n-m}-\widehat{p}_n^2=2^{2n-6}m^2\left(-6n+9m^2-18n^2-1\right)$$

Proof. Taking $b_n=\widehat{p}_n$ in Theorem 0.11 we get the required result. □

Note that for $m=1$ in Catalan's identity, we get Cassini's identity for the binomial transforms of generalized Guglielmo number. Hence, we present the corollary given below.

Corollary 0.16. *Cassini's identity for the binomial transforms of the triangular, Lucas-triangular, oblong and pentagonal numbers, respectively, are given below.*

(a) $\widehat{T}_{n-1}\widehat{T}_{n+1}-\widehat{T}^2n=-2^{2n-6}\left(2n^2+6n+8\right).$

(b) $\widehat{H}_{n+m}\widehat{H}_{n-m}-\widehat{H}_n^2=0.$

(c) $\widehat{O}_{n-1}\widehat{O}_{n+1}-\widehat{O}^2n=-2^{2n-4}\left(2n^2+6n+8\right).$

(d) $\widehat{p}_{n-1}\widehat{p}_{n+1}-\widehat{p}^2n=-2^{2n-6}\left(18n^2+6n-8\right).$

Sum Formulas

In this section, in the first instance, we give some properties that we need rest of this section and then we present some sum formulas related to binomial transform of generalized Guglielmo numbers.

0.1. *Sums of Terms with Positive Subscripts*

The following proposition can be obtained easily.

Proposition 3. *The following identities are true.*

(a) $\sum\limits_{k=0}^{n}k2^k=2^{n+1}(n-1)+2.$

(b) $\sum\limits_{k=0}^{n}k^2 2^k=2^{n+1}\left(n^2-2n+3\right)-6.$

(c) $\sum\limits_{k=0}^{n}-k2^{-k}=2^{-n}n+2^{-n+1}-2.$

(d) $\sum\limits_{k=0}^{n}k^2 2^{-k}=6-2^{-n}\left(n^2+4n+6\right).$

Theorem 0.17. *For $n \geq 0$, the following sum formulas are holds where A_1, A_2 and A_3 stated in the Theorem 0.8.*

(a) $\sum\limits_{k=0}^{n} b_k = A_2 - A_1 - A_3 + \frac{1}{4}2^{n+1}\left(4A_1 - 2A_2 + 2A_3 + 2nA_2 - nA_3 + n^2A_3\right).$

(b) $\sum\limits_{k=0}^{n} b_{2k} = -\frac{1}{27}(9A_1 - 12A_2 + 14A_3) + \frac{1}{54}2^{2n+2}\left(18A_1 - 6A_2 + 7A_3 + 18nA_2 - 3nA_3 + 18n^2A_3\right)$

(c) $\sum\limits_{k=0}^{n} b_{2k+1} = -\frac{1}{27}\left(18A_1 - 15A_2 + 13A_3\right) + \frac{1}{27}2^{2n+2}\left(18A_1 + 3A_2 + 10A_3 + 18nA_2 + 15nA_3 + 18n^2A_3\right)$

Proof. Using (0.10) and Proposition 3, the proof can be done easily.

From the Theorem 0.17, we have the following corollary for binomial transform of triangular numbers.

Corollary 0.18. *For $n \geq 0$, the following sum formulas hold.*

(a) $\sum\limits_{k=0}^{n} \widehat{T}_k = 2^{n-2}n\left(n+1\right).$

(b) $\sum\limits_{k=0}^{n} \widehat{T}_{2k} = \frac{1}{27}\left(18 \times 2^{2n}n^2 + 2^{2n} + 15 \times 2^{2n}n - 1\right).$

(c) $\sum\limits_{k=0}^{n} \widehat{T}_{2k+1} = \frac{1}{27}\left(36 \times 2^{2n}n^2 + 26 \times 2^{2n} + 66 \times 2^{2n}n + 1\right).$

From the Theorem 0.17, we have the following corollary for binomial transform of Lucas-triangular numbers.

Corollary 0.19. *For $n \geq 0$, the following sum formulas hold.*

(a) $\sum\limits_{k=0}^{n} \widehat{H}_k = 3\left(2^{n+1} - 1\right).$

(b) $\sum\limits_{k=0}^{n} \widehat{H}_{2k} = 2^{2n+2} - 1.$

(c) $\sum\limits_{k=0}^{n} \widehat{H}_{2k+1} = 2\left(2^{2n+2} - 1\right).$

From the Theorem 0.17, we have the following corollary for binomial transform of oblong numbers.

Corollary 0.20. *For $n \geq 0$, the following sum formulas hold.*

(a) $\sum\limits_{k=0}^{n} \widehat{O}_k = 2^{n-1}n\left(n+1\right).$

(b) $\sum\limits_{k=0}^{n} \widehat{O}_{2k} = \frac{2}{27}\left(18 \times 2^{2n}n^2 + 2^{2n} + 15 \times 2^{2n}n - 1\right).$

(c) $\sum\limits_{k=0}^{n} \widehat{O}_{2k+1} = \frac{2}{27}\left(36 \times 2^{2n}n^2 + 26 \times 2^{2n} + 66 \times 2^{2n}n + 1\right).$

From the Theorem 0.17, we have the following corollary for binomial transform of pentegonal numbers.

Corollary 0.21. *For $n \geq 0$, the following sum formulas hold.*

(a) $\sum\limits_{k=0}^{n} \widehat{p}_k = \frac{1}{4}\left(-5 \times 2^n n + 3 \times 2^n n^2 + 8 \times 2^n - 8\right).$

(b) $\sum\limits_{k=0}^{n} \widehat{p}_{2k} = 2^{2n+1}n^2 + 2^{2n} - 2^{2n}n - 1.$

(c) $\sum\limits_{k=0}^{n} \widehat{p}_{2k+1} = 2^{2n+2}n^2 + 2^{2n+1} + 2^{2n+1}n - 1.$

0.2. *Sums of Terms with Negative Subscripts*

The following proposition presents some formulas for the binomial transform of generalized Guglielmo numbers with negative subscripts.

Theorem 0.22. *For $n \geq 1$, the following sum formulas are holds where A_1, A_2 and A_3 stated in the Theorem 0.8.*

(a) $\sum\limits_{k=0}^{n} b_{-k} = A_1 - A_2 + A_3 - 2^{-n-2}(4A_1 - 4A_2 + 4A_3 - 2nA_2 + 3nA_3 + n^2A_3).$

(b) $\sum\limits_{k=0}^{n} b_{-2k} = \frac{1}{3}A_1 - \frac{4}{9}A_2 + \frac{14}{27}A_3 - \frac{1}{27}2^{-2n-1}(18A_1 - 24A_2 + 28A_3 - 18nA_2 + 39nA_3 + 18n^2A_3).$

(c) $\sum\limits_{k=0}^{n} b_{-2k+1} = \frac{2}{3}A_1 - \frac{5}{9}A_2 + \frac{13}{27}A_3 - \frac{1}{27}2^{-2n}(18A_1 - 15A_2 + 13A_3 - 18nA_2 + 21nA_3 + 18n^2A_3).$

Proof. The proof can be done easily using (0.10) and Proposition 3.

From the Theorem 0.22, we have the following corollary for binomial transform of triangular numbers.

Corollary 0.23. *For $n \geq 1$, the following sum formulas hold.*

(a) $\sum\limits_{k=1}^{n} T_{-k} = -2^{-n-3}n\left(n+1\right).$

(b) $\sum\limits_{k=1}^{n} T_{-2k} == -\frac{1}{108}2^{-2n}(21n - 4 \times 2^{2n} + 18n^2 + 4).$

(c) $\sum\limits_{k=1}^{n} T_{-2k+1} = -\frac{1}{54}2^{-2n}(3n + 2 \times 2^{2n} + 18n^2 - 2).$

From the Theorem 0.22, we have the following corollary for binomial transform of Lucas-triangular numbers.

Corollary 0.24. *For $n \geq 1$, the following sum formulas hold.*

(a) $\sum\limits_{k=1}^{n} H_{-k} = 3(1 - 2^{-n}).$

(b) $\sum\limits_{k=1}^{n} H_{-2k} = (1 - 2^{-2n}).$

(c) $\sum\limits_{k=1}^{n} H_{-2k+1} = 2(1 - 2^{-2n}).$

From the Theorem 0.22, we have the following corollary for binomial transform of oblong numbers.

Corollary 0.25. *For $n \geq 1$, the following sum formulas hold.*

(a) $\sum_{k=1}^{n} O_{-k} = -2^{-n-2}n\left(n+1\right).$

(b) $\sum_{k=1}^{n} O_{-2k} = -\frac{1}{54}2^{-2n}(21n - 4 \times 2^{2n} + 18n^2 + 4).$

(c) $\sum_{k=1}^{n} O_{-2k+1} = -\frac{1}{27}2^{-2n}(3n + 2 \times 2^{2n} + 18n^2 - 2).$

From the Theorem 0.22, we have the following corollary for binomial transform of pentegonal numbers.

Corollary 0.26. *For $n \geq 1$, the following sum formulas hold.*

(a) $\sum_{k=1}^{n} p_{-k} = -2^{-n-3}(11n - 16 \times 2^n + 3n^2 + 16).$

(b) $\sum_{k=1}^{n} p_{-2k} = -2^{-2n-2}(5n - 4 \times 2^{2n} + 2n^2 + 4).$

(c) $\sum_{k=1}^{n} p_{-2k+1} = -2^{-2n-1}(3n - 2 \times 2^{2n} + 2n^2 + 2).$

Matrices Related with Binomial Transform of Generalized Guglielmo Numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{0.14}$$

For matrix formulation (0.14), see [0.2].

For the binomial transforms of generalized Guglielmo numbers, we define the square matrix A of order 3 as:

$$A = \begin{pmatrix} 6 & -12 & 8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 8$. From (0.9) we have

$$\begin{pmatrix} b_{n+2} \\ b_{n+1} \\ b_n \end{pmatrix} = \begin{pmatrix} 6 & -12 & 8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_{n+1} \\ b_n \\ b_{n-1} \end{pmatrix} \tag{0.15}$$

and from (0.14) (or using (0.15) and induction) we have

$$\begin{pmatrix} b_{n+2} \\ b_{n+1} \\ b_n \end{pmatrix} = \begin{pmatrix} 6 & -12 & 8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} b_2 \\ b_1 \\ b_0 \end{pmatrix}.$$

If we take $b_n = \widehat{T}_n$ in (0.15) we have

$$\begin{pmatrix} \widehat{T}_{n+2} \\ \widehat{T}_{n+1} \\ \widehat{T}_n \end{pmatrix} = \begin{pmatrix} 6 & -12 & 8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{T}_{n+1} \\ \widehat{T}_n \\ \widehat{T}_{n-1} \end{pmatrix}. \tag{0.16}$$

For $n \geq 0$, we define

$$B_n = \begin{pmatrix} \sum_{k=0}^{n+1} \widehat{T}_k & -12\sum_{k=0}^{n} \widehat{T}_k + 8\sum_{k=0}^{n-1} \widehat{T}_k & 8\sum_{k=0}^{n} \widehat{T}_k \\ \sum_{k=0}^{n} \widehat{T}_k & -12\sum_{k=0}^{n-1} \widehat{T}_k + 8\sum_{k=0}^{n-2} \widehat{T}_k & 8\sum_{k=0}^{n-1} \widehat{T}_k \\ \sum_{k=0}^{n-1} \widehat{T}_k & -12\sum_{k=0}^{n-2} \widehat{T}_k + 8\sum_{k=0}^{n-3} \widehat{T}_k & 8\sum_{k=0}^{n-2} \widehat{T}_k \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} b_{n+1} & -12b_n + 8b_{n-1} & 8b_n \\ b_n & -12b_{n-1} + 8b_{n-2} & 8b_{n-1} \\ b_{n-1} & -12b_{n-2} + 8b_{n-3} & 8b_{n-2} \end{pmatrix}.$$

Theorem 0.27. *For all integers $m, n \geq 0$, we have*

(a) $B_n = A^n$.
(b) $C_1 A^n = A^n C_1$.
(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof**.**

(a) For the proof we use the mathematical induction on n.

First, if we take $n = 0$, the identity, (a) holds. Then, we assume that the identity (a) holds for $n = u$. Let's prove that the identity, (a) holds for $n = u + 1$. Hence, we write,

$$\begin{aligned} A^{u+1} &= \begin{pmatrix} 6 & -12 & 8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 6 & -12 & 8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^u \\ &= \begin{pmatrix} 6 & -12 & 8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B_u \\ &= \begin{pmatrix} 6 & -12 & 8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sum_{k=0}^{u+1} \widehat{T}_k & -12\sum_{k=0}^{u} \widehat{T}_k + 8\sum_{k=0}^{u-1} \widehat{T}_k & 8\sum_{k=0}^{u} \widehat{T}_k \\ \sum_{k=0}^{u} \widehat{T}_k & -12\sum_{k=0}^{u-1} \widehat{T}_k + 8\sum_{k=0}^{u-2} \widehat{T}_k & 8\sum_{k=0}^{u-1} \widehat{T}_k \\ \sum_{k=0}^{u-1} \widehat{T}_k & -12\sum_{k=0}^{u-2} \widehat{T}_k + 8\sum_{k=0}^{u-3} \widehat{T}_k & 8\sum_{k=0}^{u-2} \widehat{T}_k \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=0}^{u+1} \widehat{T}_{k+1} & -12\sum_{k=0}^{u} \widehat{T}_{k+1} + 8\sum_{k=0}^{u} \widehat{T}_k & 8\sum_{k=0}^{u} \widehat{T}_{k+1} \\ \sum_{k=0}^{u+1} \widehat{T}_k & -12\sum_{k=0}^{u} \widehat{T}_k + 8\sum_{k=0}^{u-1} \widehat{T}_k & 8\sum_{k=0}^{u} \widehat{T}_k \\ \sum_{k=0}^{u} \widehat{T}_k & -12\sum_{k=0}^{u-1} \widehat{T}_k + 8\sum_{k=0}^{u-2} \widehat{T}_k & 8\sum_{k=0}^{u-1} \widehat{T}_k \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=0}^{u+2} \widehat{T}_k & -12\sum_{k=0}^{u+1} \widehat{T}_k + 8\sum_{k=0}^{u} \widehat{T}_k & 8\sum_{k=0}^{u+1} \widehat{T}_k \\ \sum_{k=0}^{u+1} \widehat{T}_k & -12\sum_{k=0}^{u} \widehat{T}_k + 8\sum_{k=0}^{u-1} \widehat{T}_k & 8\sum_{k=0}^{u} \widehat{T}_k \\ \sum_{k=0}^{u} \widehat{T}_k & -12\sum_{k=0}^{u-1} \widehat{T}_k + 8\sum_{k=0}^{u-2} \widehat{T}_k & 8\sum_{k=0}^{u-1} \widehat{T}_k \end{pmatrix} \\ &= B_{u+1.} \end{aligned}$$

Consequently, the proof is finished by using mathematical induction on n. □

(b) Using matrix multiplication, (b) follows. □

(c) The following identity is true.

$$\begin{aligned} AC_{n-1} &= \begin{pmatrix} 6 & -12 & 8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_n & -4b_{n-1}+2b_{n-2} & 2b_{n-1} \\ b_{n-1} & -4b_{n-2}+2b_{n-3} & 2b_{n-2} \\ b_{n-2} & -4b_{n-3}+2b_{n-4} & 2b_{n-3} \end{pmatrix} \\ &= \begin{pmatrix} b_{n+1} & -4b_n+2b_{n-1} & 2b_n \\ b_n & -4b_{n-1}+2b_{n-2} & 2b_{n-1} \\ b_{n-1} & -4b_{n-2}+2b_{n-3} & 2b_{n-2} \end{pmatrix} = C_n. \end{aligned}$$

i.e. $C_n = AC_{n-1}$. If we use the induction on n and the last equation, we get $C_n = A^{n-1}C_1$. So that, the following identity is true.

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^mC_1 = A^{n-1}C_1A^m = C_nB_m$$

and the proof of $C_{n+m} = B_mC_n$ can be done similarly. □

Theorem 0.28. *Let n, m be non negative integers, then the following identities are true.*

$$\begin{aligned} b_{n+m} &= b_n \sum_{k=0}^{m+1} \widehat{T}_k + b_{n-1}\left(-12\sum_{k=0}^{m} \widehat{T}_k + 8\sum_{k=0}^{m-1} \widehat{T}_k\right) + 8b_{n-2}\sum_{k=0}^{m} \widehat{T}_k \qquad (0.17) \\ &= b_n \sum_{k=0}^{m+1} \widehat{T}_k + (-12b_{n-1} + 8b_{n-2})\sum_{k=0}^{m} \widehat{T}_k + 8b_{n-1}\sum_{k=0}^{m-1} \widehat{T}_k. \qquad (0.18) \end{aligned}$$

Proof. From the equation Theorem 0.27, (c), we see that an element of C_{n+m} is the product of row C_n and a column B_m. So, it can be easily seen that an element of C_{n+m} is the product of a row C_n and column B_m. Let $(C_{n+m})_{i,j}$ denote the entry in the i-th row j-th column of C_{n+m} and $(C_nB_m)_{i,j}$ denote the entry in the i-th row j-th column of the product C_nB_m. Then, using Theorem 0.27, (c) the following identities are true

$$(C_{n+m})_{2,1} = (C_nB_m)_{2,1},$$

This completes the proof. □

From the last Theorem, we get the following corollary.

Corollary 0.29. *For $m, n \geq 0$, we have*

(a) $\widehat{T}_{n+m} = \widehat{T}_n \sum_{k=0}^{m+1} \widehat{T}_k + \widehat{T}_{n-1}\left(-12\sum_{k=0}^{m} \widehat{T}_k + 8\sum_{k=0}^{m-1} \widehat{T}_k\right) + \widehat{T}_{n-2}\sum_{k=0}^{m} \widehat{T}_k$

(b) $\widehat{H}_{n+m} = \widehat{H}_n \sum_{k=0}^{m+1} \widehat{T}_k + \widehat{H}_{n-1}\left(-12\sum_{k=0}^{m} \widehat{T}_k + 8\sum_{k=0}^{m-1} \widehat{T}_k\right) + \widehat{H}_{n-2}\sum_{k=0}^{m} \widehat{T}_k$

(c) $\widehat{O}_{n+m} = \widehat{O}_n \sum_{k=0}^{m+1} \widehat{T}_k + \widehat{O}_{n-1}\left(-12\sum_{k=0}^{m} \widehat{T}_k + 8\sum_{k=0}^{m-1} \widehat{T}_k\right) + \widehat{O}_{n-2}\sum_{k=0}^{m} \widehat{T}_k$

(d) $\widehat{p}_{n+m} = \widehat{p}_n \sum_{k=0}^{m+1} \widehat{T}_k + \widehat{p}_{n-1}\left(-12\sum_{k=0}^{m} \widehat{T}_k + 8\sum_{k=0}^{m-1} \widehat{T}_k\right) + \widehat{p}_{n-2}\sum_{k=0}^{m} \widehat{T}_k$

Now, we consider non-positive subscript cases. For $n \geq 0$, we define

$$B_{-n} = \begin{pmatrix} -\sum_{k=0}^{n-2} \widehat{T}_{-k} & 12\sum_{k=0}^{n-1} \widehat{T}_{-k} - 8\sum_{k=0}^{n} \widehat{T}_{-k} & -8\sum_{k=0}^{n-1} \widehat{T}_{-k} \\ -\sum_{k=0}^{n-1} \widehat{T}_{-k} & 12\sum_{k=0}^{n} \widehat{T}_{-k} - 8\sum_{k=0}^{n+1} \widehat{T}_{-k}) & -8\sum_{k=0}^{n} \widehat{T}_{-k} \\ -\sum_{k=0}^{n} \widehat{T}_{-k} & 12\sum_{k=0}^{n+1} \widehat{T}_{-k} - 8\sum_{k=0}^{n+2} \widehat{T}_{-k}) & -8\sum_{k=0}^{n+1} \widehat{T}_{-k} \end{pmatrix}$$

and

$$C_{-n} = \begin{pmatrix} b_{-n+1} & -12b_{-n} + 8b_{-n-1} & 8b_{-n} \\ b_{-n} & -12b_{-n-1} + 8b_{-n-2} & 8b_{-n-1} \\ b_{-n-1} & -12b_{-n-2} + 8b_{-n-3} & 8b_{-n-2} \end{pmatrix}.$$

By convention, we assume that

$$\sum_{k=0}^{-1} \widehat{T}_{-k} = 0 \text{ , and } \sum_{k=0}^{-2} \widehat{T}_{-k} = 1$$

Theorem 0.30. *For all integers $m, n \geq 0$, we have*

(a) $B_{-n} = A^{-n}$.
(b) $C_{-1}A^{-n} = A^{-n}C_{-1}$.
(c) $C_{-n-m} = C_{-n}B_{-m} = B_{-m}C_{-n}$.

Proof.

(a) Using mathematical induction on n,the proof can be done easily .
(b) Using matrix multiplication, (b) follows.
(c) The following identity is true.

$$\begin{aligned} AC_{-n-1} &= \begin{pmatrix} 6 & -12 & 8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_{-n} & -12b_{-n-1} + 8b_{-n-2} & 8b_{-n-1} \\ b_{-n-1} & -12b_{-n-2} + 8b_{-n-3} & 8b_{-n-2} \\ b_{-n-2} & -12b_{-n-3} + 8b_{-n-4} & 8b_{-n-3} \end{pmatrix} \\ &= \begin{pmatrix} b_{-n+1} & -12b_{-n} + 8b_{-n-1} & 8b_{-n} \\ b_{-n} & -12b_{-n-1} + 8b_{-n-2} & 8b_{-n-1} \\ b_{-n-1} & -12b_{-n-2} + 8b_{-n-3} & 8b_{-n-2} \end{pmatrix} = C_{-n}, \end{aligned}$$

i.e. $C_{-n} = AC_{-n-1}$. From the last equation, using induction, we obtain $C_{-n} = A^{-n+1}C_{-1}$. Now,

$$C_{-n-m} = A^{-n-m+1}C_{-1} = A^{-n+1}A^{-m}C_{-1} = A^{-n+1}C_{-1}A^{-m} = C_{-n}B_{-m}$$

and the proof of $C_{-n-m} = B_{-m}C_{-n}$ can be done similarly.□

Theorem 0.31. *For $m, n \geq 0$, we have*

$$\begin{aligned} b_{-n-m} &= -b_{-n+1}\sum_{k=0}^{m-1} T_{-k} - b_{-n}\left(-12\sum_{k=0}^{m} T_{-k} + 8\sum_{k=0}^{m+1} T_{-k}\right) - 8b_{-n-1}\sum_{k=0}^{m} T_{-k} \\ &= -b_{-n}\sum_{k=0}^{m-2} \widehat{T}_{-k} - (-12b_{-n-1} + 8b_{-n-2})\sum_{k=0}^{m-1} \widehat{T}_{-k} - 8b_{-n-1}\sum_{k=0}^{m} \widehat{T}_{-k}. \end{aligned}$$

Proof. From the Theorem 0.30, (c), we see that an element of C_{-n-m} is the product of row C_{-n} and a column B_{-m}. So, it can be easily seen that an element of C_{-n-m} is the product of a row C_{-n} and column B_{-m}. Let $(C_{-n-m})_{i,j}$ denote the entry in the i-th row j-th column of C_{-n-m} and $(C_{-n}B_{-m})_{i,j}$ denote the entry in the i-th row j-th column of the product $C_{-n}B_{-m}$. Then, using Theorem 0.27, (c) the following identities are true

$$(C_{-n-m})_{2,1} = (C_{-n}B_{-m})_{2,1}.$$

This completes the proof. □

Corollary 0.32. *For $m, n \geq 0$, we have*

(a) $\widehat{T}_{-n-m} = -\widehat{T}_{-n}\sum_{k=0}^{m-2}\widehat{T}_{-k} - \widehat{T}_{-n-1}\left(-12\sum_{k=0}^{m-1}\widehat{T}_{-k} + 8\sum_{k=0}^{m}\widehat{T}_{-k}\right) - 8\widehat{T}_{-n-2}\sum_{k=0}^{m-1}\widehat{T}_{-k}$.

(b) $\widehat{H}_{-n-m} = -\widehat{H}_{-n}\sum_{k=0}^{m-2}\widehat{T}_{-k} - \widehat{H}_{-n-1}\left(-12\sum_{k=0}^{m-1}\widehat{T}_{-k} + 8\sum_{k=0}^{m}\widehat{T}_{-k}\right) - 8\widehat{H}_{-n-2}\sum_{k=0}^{m-1}\widehat{T}_{-k}$.

(c) $\widehat{O}_{-n-m} = -\widehat{O}_{-n}\sum_{k=0}^{m-2}\widehat{T}_{-k} - \widehat{O}_{-n-1}\left(-12\sum_{k=0}^{m-1}\widehat{T}_{-k} + 8\sum_{k=0}^{m}\widehat{T}_{-k}\right) - 8\widehat{O}_{-n-2}\sum_{k=0}^{m-1}\widehat{T}_{-k}$.

(d) $\widehat{p}_{-n-m} = -\widehat{p}_{-n}\sum_{k=0}^{m-2}\widehat{T}_{-k} - \widehat{p}_{-n-1}\left(-12\sum_{k=0}^{m-1}\widehat{T}_{-k} + 8\sum_{k=0}^{m}\widehat{T}_{-k}\right) - 8\widehat{p}_{-n-2}\sum_{k=0}^{m-1}\widehat{T}_{-k}$.

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